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ON THE OPTIMAL DESIGN OF MULTI-SPEED GEAR TRAINS

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Abstract—An efficient method for analyzing multi-speed gear trains is presented and used in this paper. This new technique obtains equations for the diameters of all the gears used in the transmission based on information contained in the speed diagram. The form of the equations is such that they can be generated by the computer automatically. Further, the equations are applicable to the general arrangement, the single composite arrangements, and the double composite arrangements. As a result, all of the promising kinematic arrangements possible for a given number of speeds can be easily studied. By promoting different constraints and objective functions, trace-offs between different parameters can easily be examined. The technique is illustrated using a case study of a 9 speed gear train. A multi-parameter optimization technique is used to solve 19 different arrangements for a weighted objective function minimizing volume and maximizing stiffness.

1. INTRODUCTION

The purpose of a multi-speed gear train is to provide a set of output speeds that meet the user's requirements at a reasonable cost. The designer must minimize the cost and volume while maximizing it's strength.

The selection of the kinematic arrangement is an important step in the design process of gear trains if the above objectives are to be met. For a given number of speeds there are usually a number of speed diagrams and gear arrangements that can produce the required output speeds in a geometric progression thus satisfying the basic kinematic requirements[1–3]. Not all arrangements are equally satisfactory. Some arrangements are better because they use fewer gears, or fewer shafts, or cover a larger speed range, etc. Also, the kinematic arragement affects the dynamic properties of the gear train. Marchelek[4] has shown that an arrangement avoiding excessive gear ratios will greatly improve the static stiffness of the gear train assuring that vibration will not reduce it's effectiveness.

Selecting the best kinematic arrangement, however, is not obvious. There are many possible arrangements which will produce a specific number of speeds. For example, Fig. 1 shows that a four speed gear train having a geometric progression of output speeds can be obtained using 2 shafts and 8 gears, 3 shafts and 8 gears, 3 shafts and 7 gears, or 3 shafts and 6 gears. Each arrangement has certain advantages and disadvantages. The layouts in Figs. 1(c, d) are known as single composite and double composite arrangements, respectively.

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Composite arrangements have at least one gear on an intermediate shaft that acts as a driven gear and a driving pinion at the same time. If only one of the gears on the intermediate shaft has this property, the layout is termed a "single composite". If the 2 gears on the same intermediate shaft have this property, the layout is termed a "double composite". Similarly, an arrangement having 3 gears on the same intermediate shaft would be termed a "triple composite". Hall[5] and White[6] have shown that triple composite arrangements are unable to produce output speeds in a geometric progression and hence will not be considered. Composite gear arrangements reduce the number of gears required to produce a given number of speeds since the function of two gears on the intermediate shaft is done by a single gear. This makes composite arrangements attractive to gear train designers. Unfortunately, it increases the number of arrangements for a given number of output speeds and makes many of the gear diameters interdependent necessitating the development of mathematical models and techniques to produce feasible solutions[7–10].

The 9 speed arrangement shown in Fig. 2 has 1 conventional arrangement, 9 single composite arrangements and 9 double composite arrangements. It also has 2 different speed diagrams, Fig. 3, thus, 38 mathematically different arrangements are possible. As the number of speeds increase, the number of possible arrangements tends to increase as shown in Table 1.

Most mathematical models are based on the speed diagram. The speeds of each shaft are plotted on a logarithmic scale. This way the relative slope of the lines is proportional to the gear ratios and the spacing between them is constant since the output speeds from a geometric progression.

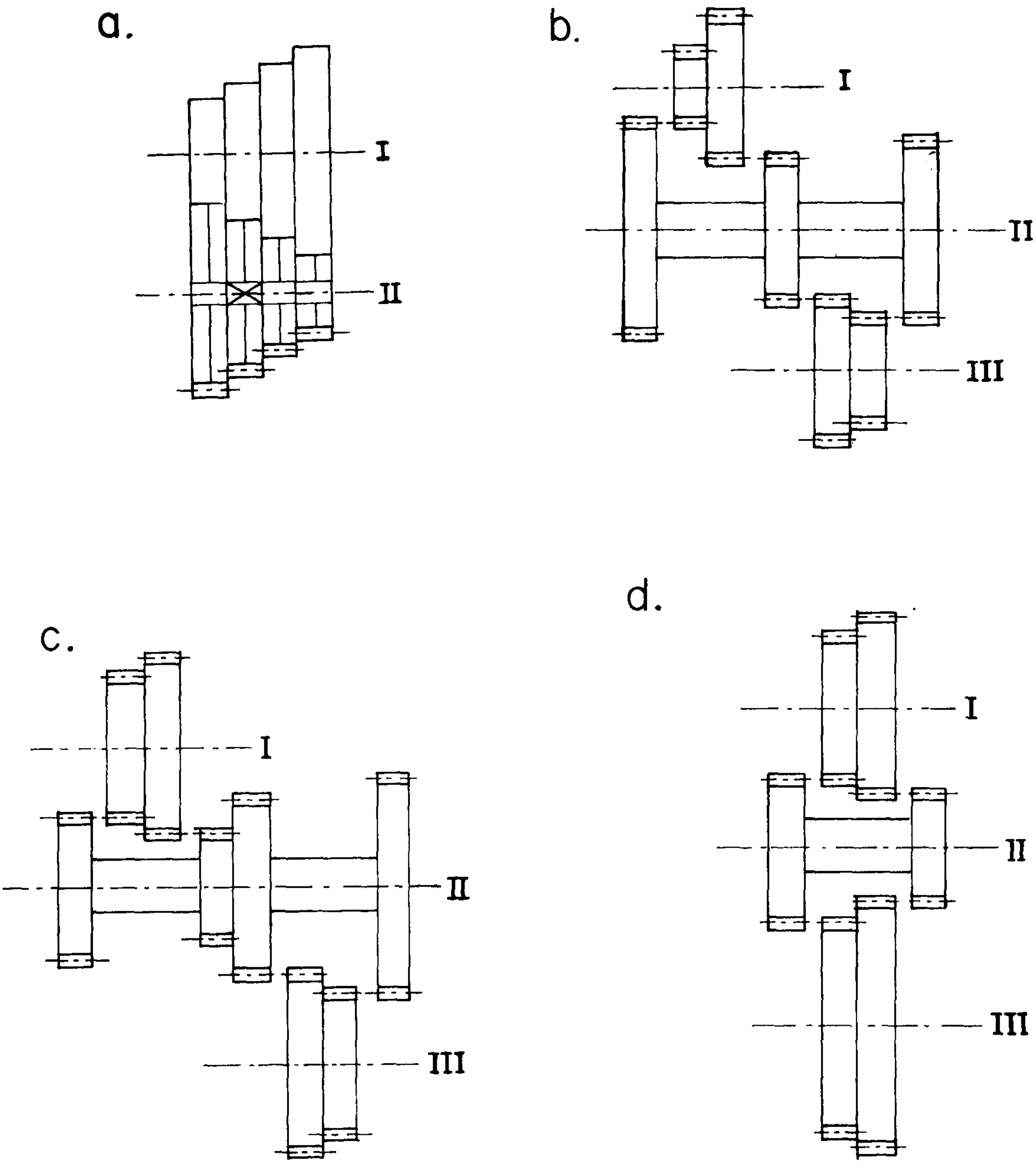
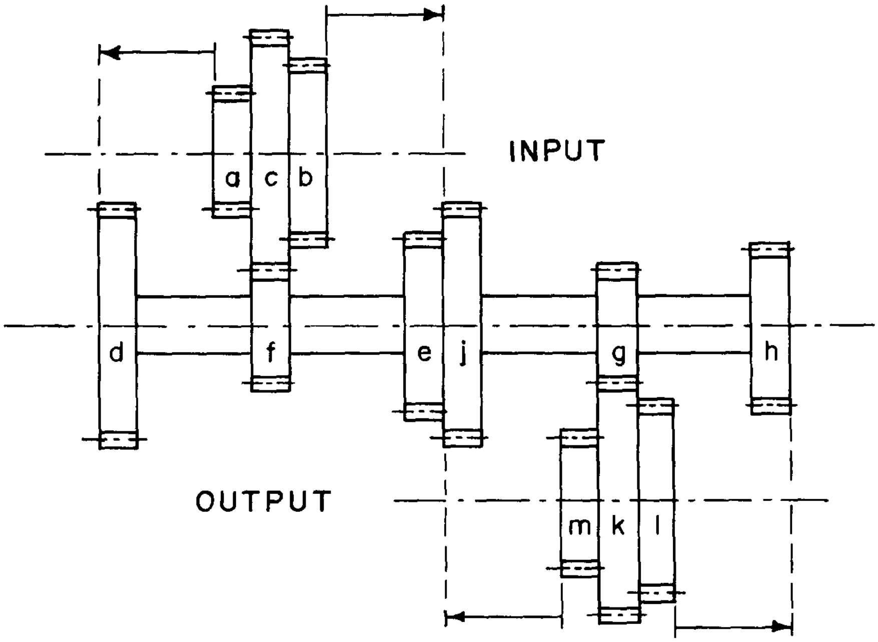


Fig. 1. Possible 4 speed layouts using: (a) 2 shafts = 8 gears, (b) 3 shafts = 8 gears, (c) 3 shafts = 7 gears, (d) 3 shafts = 6 gears.

Hall, White, White and Sanger and Sanger and White developed analytic expressions for the gear diameters of specific double composite arrangements. Once analytic expression were available it was possible to optimize a gear train based on kinematic parameters[11, 12].

Murthy[13] studied single composite gear trains in an effort to optimize the radial dimensions. He found that in particular cases the single composite arrangement was as small as the conventional gear arrangements. His technique did not develop equations for the diameters of each of the gears in a general form.

Osman *et al.*[12] studied the 9 speed–10 gear transmission using a multi-parameter optimization technique which allowed the equations for the gear diameters to be written in a simpler form. This



Special Cases

	Single Composite	Double Composite	
	Condition	Conditions	Type(s)
1.	d = g	d = j ; e = g	XI
2.	d = h	d = j ; e = h	VIII
3.	d = j	d = j ; f = g	IX, XII
4.	e = g	d = j ; f = h	V
5.	e = h	d = h ; e = g	I, VI
6.	e = j	d = h ; f = g	III
7.	f = g	e = j ; f = g	VII
8.	f = h	e = j ; f = h	IV
9.	f = j	e = h ; f = g	II, X

Fig. 2. Conventional 9 speed gear train showing how each of the 18 composite arrangements may be obtained.

reduced much of the work required to obtain solutions to different objective functions.

Previous designers have concentrated their efforts on developing gear diameter equations for the single and double composite gear train arrangements because they offered the most savings in gears. As a result, general diameter equations have applied primarily to double composite arrangements and have been unnecessarily complicated. Single and double composite gear trains are special cases of the conventional gear arrangement, as shown in Fig. 2. Hence a general theory for conventional gear arrangements should be applicable to single composite and double composite arrangements as well.

This paper presents a method which allows the designer to study each of the promising arrangements. All of the possible arrangements (conventional, single composite, and double composite) may be formulated from one set of equations based on information supplied by the speed diagram, a user supplied objective function and user supplied constraints. Since the speed diagrams may be formulated automatically by a computer, and since the equations may be written from the speed diagram, the entire

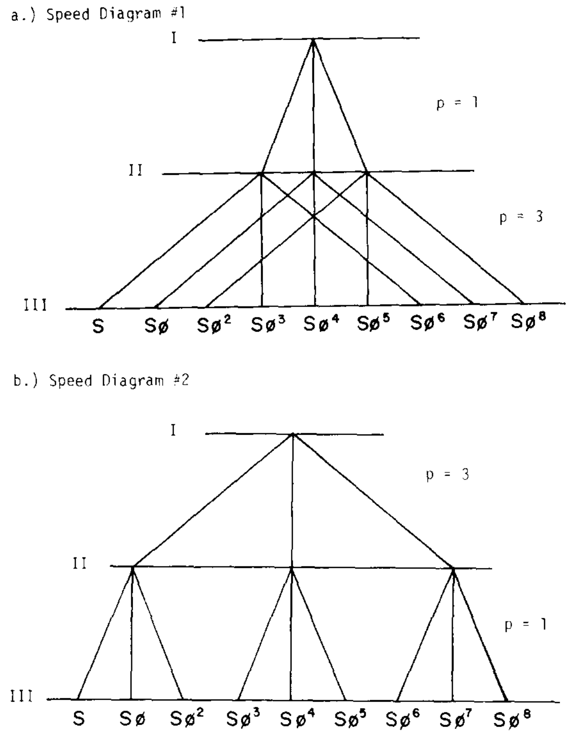


Fig. 3. Possible speed diagrams for a nine speed gear train using 3 shafts.

process can be computerized, saving the designer considerable time.

2. THE MATHEMATICAL MODEL

The simplest case of a multi-speed gear train consists of two shafts with two or more gear pairs. This simple gear train is shown in Fig. 4 with it's speed diagram.

In order for the shaft distance to be the same for all the meshing gear pairs on common shafts the following equations must hold

$$a + e = b + f \tag{1}$$

$$a + e = c + g \tag{2}$$

$$a + e = d + h. \tag{3}$$

Also required is that the output speeds form a geometric progression with the lowest output/input speed ratio being S. This gives

$$S = a/e \tag{4}$$

$$S\phi = b/f \tag{5}$$

$$S\phi^2 = c/g \tag{6}$$

$$S\phi^3 = d/h. \tag{7}$$

Table 1. Number of arrangement possible for a given number of speeds

Number of Speeds	Factors	General Arrangements	Single Composite	Double Composite	Sub Total	Total
3	3-1	1	-	-	1	1
4	4-1	1	-	-	1	
	2-2	1	4	1	6	7
5	5-1	1	-	-	1	1
6	6-1	1	-	-	1	
	2-3*	1	6	3	10	11
8	8-1	1	-	-	1	
	4-2*	1	8	6	15	
	2-2-2**	1	8	2	11	27
9	9-1	1	-	-	1	
	3-3	1	9	9	19	20
12	12-1	1	-	-	1	
	4-3*	1	12	18	31	
	2-3-2***	1	12	4	17	48

* Rearranging the factors produces mathematically different arrangements. (i.e. different speed diagrams).

** With 2 intermediate shafts it is possible to have single and double composite arrangements at the same time introducing additional feasible arrangements.

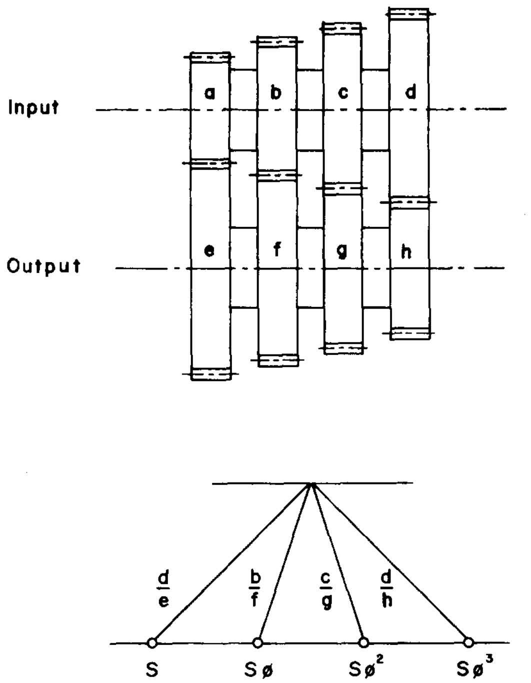


Fig. 4. Schematic of 2 shaft gear train with it's speed diagram.

These 7 equations contain 9 unknowns (a–h, S). Thus this arrangement has 2 degrees of freedom. All 2 shaft arrangements, regardless of the number of gear pairs have 2 degrees of freedom after satisfying mesh and output speed requirements.

If a second 2 shaft arrangement was attached to the output shaft of the first arrangement 2 more degrees of freedom would become available. Combining the two groups would result in a conventional 3 shaft arrangement. Thus, each shaft that is added and used in a conventional arrangement manner adds 2 degrees of freedom to the problem. This means it is simple to determine the number of degrees of freedom f knowing the number of shafts s for a conventional arrangement using:

$$f = (2s - 1). \tag{8}$$

Composite gear arrangements are obtained from conventional gear trains by equating a driven gear with a driving gear on the same shaft (intermediate shafts have both driven and driving gears). Each composite gear introduces 1 additional equality constraint. Mathematically an equality constraint reduces the degrees of freedom by 1. Therefore, a single composite arrangement reduces the degrees of freedom by 1 while a double composite arrangement reduces the degrees of freedom by 2.

Returning to the 2 shaft arrangement of Fig. 4, it is known now that there are 2 degrees of freedom. Thus the diameters of all the gears can be written as functions of the independent variables which are chosen to be the diameters of gears a and e (the first gear on each shaft). Letting

$$Y_i = a \tag{9}$$

$$Y_0 = e. \tag{10}$$

Equations (1)–(7) can be rewritten now as

$$Y_i + Y_0 = b + f \tag{11}$$

$$Y_i + Y_0 = c + g \tag{12}$$

$$Y_i + Y_0 = d + h \tag{13}$$

$$S = Y_i / Y_0 \tag{14}$$

$$S\phi = b/f \tag{15}$$

$$S\phi^2 = c/g \tag{16}$$

$$S\phi^3 = d/h. \tag{17}$$

Using eqns (11)–(17) the diameters of the remaining gears can be found as functions of Y_i, Y_0 and ϕ.

$$b = \frac{Y_i\phi(Y_i + Y_0)}{(Y_i\phi + Y_0)} \tag{18}$$

$$c = \frac{Y_i\phi^2(Y_i + Y_0)}{(Y_i\phi^2 + Y_0)} \tag{19}$$

$$d = \frac{Y_i\phi^3(Y_i + Y_0)}{(Y_i\phi^3 + Y_0)} \tag{20}$$

$$f = \frac{Y_0(Y_i + Y_0)}{(Y_i\phi + Y_0)} \tag{21}$$

$$g = \frac{Y_0(Y_i + Y_0)}{(Y_i\phi^2 + Y_0)} \tag{22}$$

$$h = \frac{Y_0(Y_i + Y_0)}{(Y_i\phi^3 + Y_0)}. \tag{23}$$

Comparing equations of the gears on the input shaft (a–d), the general form for the diameter of the driving gear is seen to be

$$d_{\text{pinion}} = \frac{Y_i\phi^{l-1}(Y_i + Y_0)}{(Y_i\phi^{l-1} + Y_0)} \quad l = 1, 2, \ldots, w \tag{24}$$

where w is the number of gears on the input shaft. Comparing equations for the diameters of the driven gears (e–h)

$$d_{\text{gear}} = \frac{Y_0(Y_i + Y_0)}{(Y_i\phi^{l-1} + Y_0)} \quad l = 1, 2, \ldots, w. \tag{25}$$

Using the general input and output diameter equations the gear ratios can also be calculated. The train value is defined as

$$e = \frac{\text{Product of driving tooth numbers}}{\text{Product of driven tooth numbers}}. \tag{26}$$

The number of teeth N, is related to the pitch

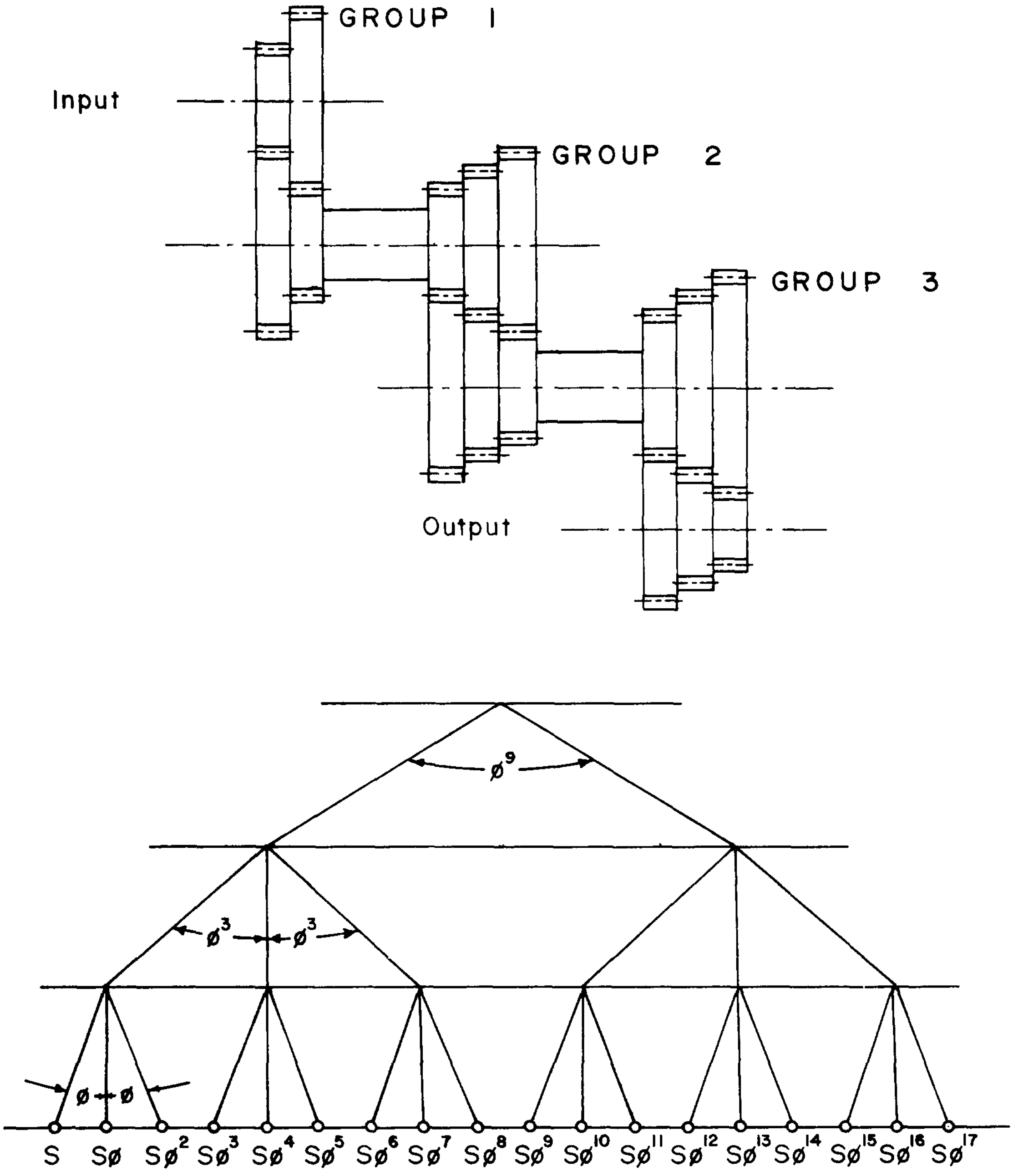


Fig. 5. Schematic of 18 speed gear train with it's speed diagram.

diameter d and the diametral pitch P as follows

$$N = dP \tag{27}$$

Since P must be the same for two meshing gears the train value between gear pairs is given as

$$e = \frac{d_{\text{pinion}}}{d_{\text{gear}}} \quad l = 1, 2, \ldots, w \tag{28}$$

Substituting eqns (25) and (26) into eqn (29) and reducing

$$e = \frac{Y_l \phi^{l-1}}{Y_0} \quad l = 1, 2, \ldots, w. \tag{29}$$

Thus the gear ratios are also defined by the same independent parameters.

These 2 shaft equations can be applied to the general multi-speed gear train if the required arrangement is separated into a combination of 2 shaft arrangements as shown in Fig. 5. This figure shows how 2 shaft arrangements can be isolated in order to write the diameter equations. The previous 2 shaft equations apply to these individual parts with the following variation.

In the two-shaft arrangement used to develop the general form of the equations the speeds increased on the output shaft by a constant factor of ϕ (Fig. 4). With multi-shaft arrangements intermediate shafts have speed increases of a constant factor ϕ^p where p is an integer designated as the range exponent. The range exponent can easily be obtained from the speed diagram either mathematically or visually. For example, the 18 speed gear train shown in Fig. 5 has two speeds in group one. From the speed diagram one sees that the second speed is greater than the first speed by a factor of ϕ^9 (the spacing between speeds is 9 times greater than that for the output shaft with speed increases of ϕ). Hence p for group 1 is 9. Group 2 has 3 speeds. Again referring to the speed diagram it is seen that the second speed is greater than the first speed by a factor of ϕ^3 and the third speed is greater than the second speed by a factor of ϕ^3 also. Thus p for group 2 is 3. Similarly, p for group 3 is 1 since each speed in the group is greater than the previous speed by the factor of ϕ^1.

This range exponent p must be included in eqns (24), (25) and (29) in order to be generally applicable. To include p, ϕ in the previous equations must be replaced by ϕ^p giving the final form of the diameter equations as:

$$d_{\text{pinion}} = \frac{Y_i\phi^{p(l-1)}(Y_i + Y_0)}{(Y_i\phi^{p(l-1)} + Y_0)} \tag{30}$$

$$d_{\text{gear}} = \frac{Y_0(Y_i + Y_0)}{(Y_i\phi^{p(l-1)} + Y_0)} \qquad l = 1, 2, \ldots, w \tag{31}$$

$$e_l = \frac{Y_i\phi^{p(l-1)}}{Y_0} \tag{32}$$

where w is the number of gear pairs in the group, p is range exponent obtained from the speed diagram, Y_i is diameter of the first gear on the input shaft, Y_0 is diameter of the first gear on the output shaft, and e_l is the gear ratio for the lth pair.

Given the speed diagram one can directly write the equations for the diameters of the gears relating to the conventional arrangement using eqns (30) and (31). Single composite arrangements are obtained by equating equations for 1 gear and 1 pinion on the same intermediate shaft. Double composite arrangements are obtained by equating 2 gears with 2 pinions on the same intermediate shaft. When pairing gears for double composite arrangements it should be noted that not all pairs are feasible. For example, if the first gear is larger than the second gear then the pinions must be selected such that the first pinion is smaller than the second pinion.

3. A CASE STUDY

The 9 speed–10 gear arrangement can be used to illustrate how diameter equations for each of the gears can be obtained using the general equations. Figure 6 shows the schematic layout of the conventional 9 speed gear train using 3 shafts along with one of it's speed diagram. The following procedure should be used:

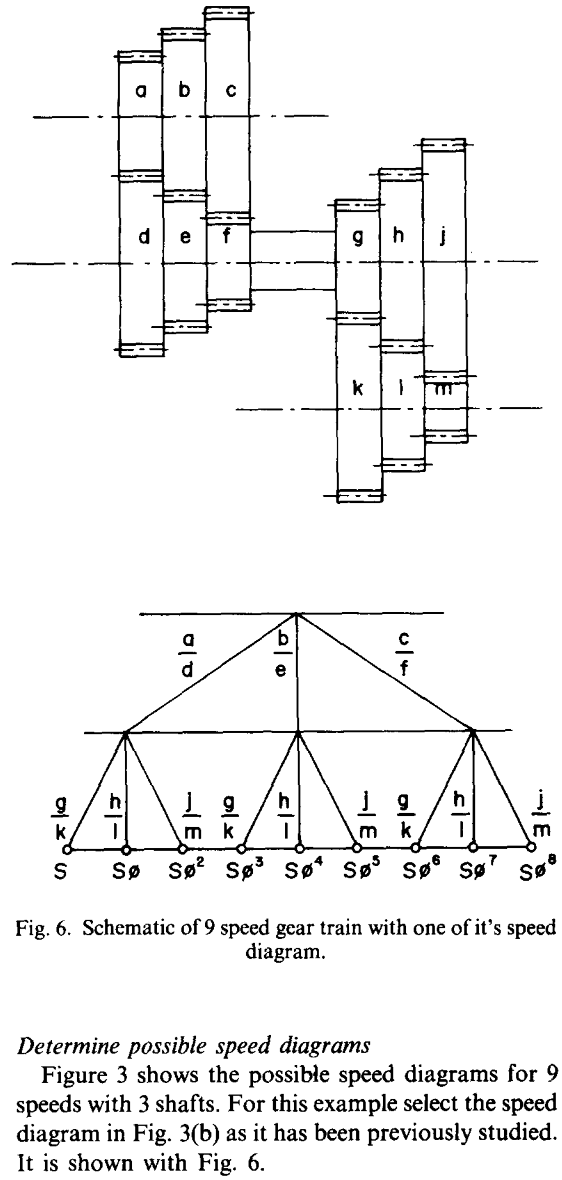


Fig. 6. Schematic of 9 speed gear train with one of it's speed diagram.

Determine possible speed diagrams

Figure 3 shows the possible speed diagrams for 9 speeds with 3 shafts. For this example select the speed diagram in Fig. 3(b) as it has been previously studied. It is shown with Fig. 6.

Draw schematic

Using the information contained in the speed diagram, draw the conventional layout as shown in Fig. 6 (even if composite arrangements only are of interest). The information obtained from the speed diagram at this stage is the number of speeds, the number of shafts, the number of groups and the number of gears in each group.

Label schematic

The general equations yield pinion diameters in increasing order and gear diameters in decreasing order because of the way in which they were formulated. Therefore label, using lower case letters, the pinions in order of increasing diameter and the gears in order of decreasing diameter.

Divide the gear train into groups

From Fig. 6, it is seen that this gear train has 2 groups. Group 1 includes gears "a"–"f" and relates to the upper part of the speed diagram. Group 2 includes gears "g"–"m" and relates to the lower part of the speed diagram.

Find the parameters for each group

The number of gears pairs in the first group w is equal to 3. The range exponent p is obtained from the speed diagram and is 3 since the ratio b/e is greater than the ratio of a/d by ϕ^3. The first gear on the input shaft Y_i is equal to a and the first gear on the output shaft Y_0 is equal to d., Often the first pinion in the first group is known as the reference gear and is set equal to 1. Rather than using gear diameter letters such as "a' and "d" to identify the independent parameters, a vector Y is used such that $Y_1 = a$ and $Y_2 = d$.

The number of gear pairs w in the second group is 3. The range exponent is 1 since the speeds increase by a factor of ϕ. $Y_3 = g$ and $Y_4 = k$.

Write general gear diameter and gear ratio equations

Using the parameters obtained above, write equations applicable to the 9 speed conventional arrangement. This gives the following set of equations normalized with respect to "a".

For group 1

Using Pinion Equation.

$$\frac{a}{a} = 1 \tag{33}$$

$$\frac{b}{a} = \frac{\phi^3(1 + Y_2)}{(\phi^3 + Y_2)} \tag{34}$$

$$\frac{c}{a} = \frac{\phi^6(1 + Y_2)}{(\phi^6 + Y_2)} \tag{35}$$

Using Gear Equation.

$$\frac{d}{a} = Y \tag{36}$$

$$\frac{e}{a} = \frac{Y_2(1 + Y_2)}{(\phi^3 + Y_2)} \tag{37}$$

$$\frac{f}{a} = \frac{Y_2(1 + Y_2)}{(\phi^6 + Y_2)} \tag{38}$$

For Group 2

Using Pinion Equation.

$$\frac{g}{a} = Y_3 \tag{39}$$

$$\frac{h}{a} = \frac{Y_3\phi(Y_3 + Y_4)}{(Y_3\phi^2 + Y_4)} \tag{40}$$

$$\frac{j}{a} = \frac{Y_3\phi^2(Y_3 + Y_4)}{(Y_3\phi^2 + Y_4)} \tag{41}$$

Using Gear Equation.

$$\frac{k}{a} = Y_4$$

$$\frac{1}{a} = \frac{Y_4(Y_3 + Y_4)}{(Y_3\phi + Y_4)} \tag{43}$$

$$\frac{m}{a} = \frac{Y_4(Y_3 + Y_4)}{(Y_3\phi^2 + Y_4)} \tag{44}$$

Equation (30) can also be applied at this point if the gear ratios are of interest.

Isolate composite arrangement of interest

Composite gear arrangements can be formed from the conventional gear arrangements by forcing pinion(s) to have the same diameter as gear(s) on the same intermediate shaft. Figure 2 shows the conventional 9 speed gear train with the conditions necessary for each of the single and double composite arrangements.

Equations (33)–(44) can be optimized using a constrained multi-parameter optimization technique [14, 15] as demonstrated by osman, Sankar and and Dukkipati[9], to provide solutions to the conventional arrangement, the 9 single composite arrangements, and the 9 double composite arrangements by providing an objective function and constraints.

For example, to generate solutions for the type XI double composite arrangement one would write an objective function first. Next, from Fig. 2, the conditions $d = j$ and $e = g$ would be noted. This corresponds to equating eqn (36) with eqn (41) and equating eqn (37) with eqn (39). Thus the problem requires the solution of an objective function subject to 2 equality constraints. In all solutions the diameter of "d" would equal the diameter of "j" and the diameter of "e" would equal the diameter of "g" . Since these gears are all on the same shaft, two of the gears would be redundent and could be eliminated.

4. SOLUTIONS USING THE COMPUTER

While the equations are very simple to develop there are still many calculations required to obtain optimum solutions. As the number of speeds increase, the number of arrangements possible increases resulting in increased calculations. The only practical solution is to utilize computers. Thus, all practical arrangements for a given number of different objective functions and constraints can be studied.

The equations developed in this paper may be generated automatically by the computer since they are based on the speed diagram which is based on geometric constraints[1]. Thus the designer needs only to develop specific functions designed to solve specific gear train problems.

A computer program was developed that would automatically find all possible speed diagrams for a given number of speeds, develop the constraints which defined the different composite arrangements and optimized each of the posssible combinations based on a user supplied objective function. The flow chart is shown in Fig. 7.

The computer program requires as inputs the number of speeds, the step ratio, an objective function, constraints developed by the designer to product feasible solutions and a number of optimization parameters.

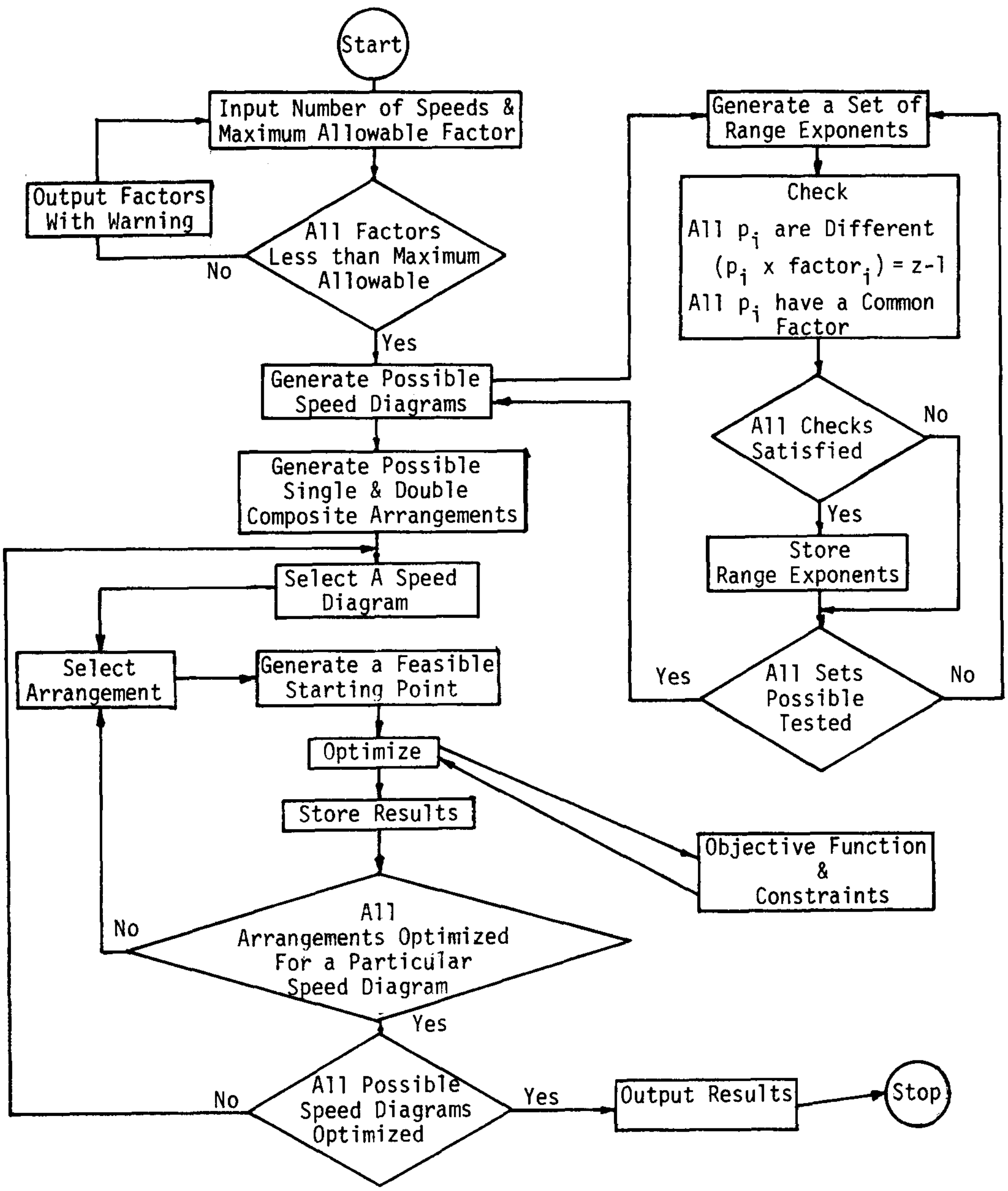


Fig. 7. Program flow chart to optimize multi-speed gear train.

5. OPTIMAL RADIAL DIMENSIONS

The 9 speed gear train was chosen as a sample problem with the step ratio taken as 1.414. The objective function was developed to minimize the radial dimensions of the gear box. The value of the objective function was set equal to the sum of the distance between the shafts plus the radii of the largest gears on the input and output shafts all normalized with respect to the smallest gear.

The constraints developed to illustrate the method were chosen to avoid generating unfeasible solutions such as negative gear diameters or excessive gear ratios. All gear diameters were forced to be positive and the gear ratios were set to be larger than 1/6. This eliminated some of the very poor double composite arrangements without using excessive computer time.

Results for the 9 speed gear train are given in Table 2. The leftmost column designates the arrangement optimized. The rightmost column gives the value of the objective function. The smaller the value of the objective function the better the solutions. The columns in the middle give the normalized gear diameters. Each row represents an optimal solution to one of the possible 9 speed arrangements. The arrangement is identified by the optimization code and Figs. (2) and (3). The first digit represents the speed

Table 2. Optimal radial dimension

Optimization	a	b	c	d	e	f	g	h	j	k	l	m	Function
1-C	1.000	1.207	1.414	1.414	1.207	1.000	1.000	1.914	2.827	2.827	1.914	1.000	5.2411
1-SC-1	1.000	1.207	1.414	1.414	1.207	1.000	1.414	2.555	3.575	3.161	2.020	1.000	5.7822
1-SC-2	1.000	1.238	1.489	1.914	1.676	1.425	1.000	1.914	2.827	2.827	1.914	1.000	5.5283
1-SC-3	1.000	1.276	1.585	2.827	2.551	2.242	1.000	1.914	2.827	2.827	1.914	1.000	6.0334
1-SC-4	1.030	1.240	1.450	1.420	1.210	1.000	1.210	2.244	3.216	3.006	1.972	1.000	5.5611
1-SC-5	1.000	1.250	1.519	2.164	1.914	1.645	1.000	1.914	2.827	2.827	1.914	1.000	5.6689
1-SC-6	1.000	1.285	1.609	3.112	2.827	2.503	1.000	1.914	2.827	2.827	1.914	1.000	6.1873
1-SC-7	1.000	1.207	1.414	1.414	1.207	1.000	1.000	1.914	2.827	2.827	1.914	1.000	5.2411
1-SC-8	1.000	1.263	1.552	2.465	2.202	1.914	1.000	1.914	2.827	2.827	1.914	1.000	5.8358
1-SC-9	1.000	1.294	1.633	3.461	3.167	2.827	1.000	1.914	2.827	2.827	1.914	1.000	6.3741
1-DC-6	5.513	6.940	8.496	13.013	11.586	10.030	10.030	13.013	14.543	5.513	2.530	1.000	24.0385
1-DC-2	5.637	7.185	8.918	15.729	14.180	12.447	11.092	14,180	15.729	5.637	2.549	1.000	26.3247
1-DC-1	2.205	2.761	3.359	4.844	4.289	3.690	2.205	3.690	4.844	3.639	2.154	1.000	9.9457
1-DC-8	5.058	6.385	7.840	12.380	11.053	9.598	6.995	9.598	11.053	5.058	2.445	1.000	21.1946
2-C	1.000	1.914	2.827	2.827	1.914	1.000	1.000	1.207	1.414	1.414	1.207	1.000	5.2411
2-SC-1	1.000	1.914	2.827	2.827	1.914	1.000	2.827	2.167	2.461	1.633	1.294	1.000	6.3742
2-SC-2	1.000	1.914	2.827	2.827	1.914	1.000	2.503	2.827	3.112	1.609	1.285	1.000	6.1873
2-SC-3	1.000	1.914	2.827	2.827	1.914	1.000	2.242	2.551	2.827	1.585	1.276	1.000	6.0334
2-SC-4	1.025	1.953	2.874	2.849	1.921	1.000	1.921	2.210	2.474	1.553	1.263	1.000	5.8872
2-SC-5	1.000	1.914	2.827	2.827	1.914	1.000	1.645	1.914	2.164	1.520	1.250	1.000	5.6689
2-SC-6	1.000	1.914	2.827	2.827	1.914	1.000	1.425	1.676	1.914	1.489	1.238	1.000	5.5284
2-SC-7	1.000	1.914	2.827	2.827	1.914	1.000	1.000	1.207	1.414	1.414	1.207	1.000	5.2411
2-SC-8	1.000	1.972	3.004	3.211	2.240	1.207	1.000	1.207	1.414	1.414	1.207	1.000	5.5217
2-SC-9	1.000	2.020	3.161	3.575	2.555	1.414	1.000	1.207	1.414	1.414	1.207	1.000	5.7822
2-DC-5	1.000	2.283	4.180	6.667	5.384	3.487	5.384	6.667	8.018	10.327	9.044	7.693	18.9423
2-DC-1	1.000	1.914	2.827	2.827	1.914	1.000	1.914	2.360	2.827	3.499	3.053	2.586	7.7833
2-DC-2	1.000	2.283	4.180	6.667	5.384	3.487	4.323	5.384	6.667	11.479	10.327	9.044	19.5181
2-DC-7	1.000	2.146	3.608	4.755	3.609	2.146	2.146	2.814	3.609	9.193	8.525	7.730	14.9471

Optimization Number - 1st digit refers to speed diagram (figure 3)
- letters refer to arrangement (C - Conventional, SC - Single Composite, and DC - Double Composite)
- last digit specifies conditions as shown in figure (8).

diagram. A "1" represents the speed diagram in Fig. 3(a), a "2" represents the speed diagram in Fig. 3(b). The letters designate the number of composite gears. The letter C means a conventional arrangement, the letters SC mean a single composite arrangement (i.e. 2 gears on the intermediate shaft must have the same diameter) and the letters DC mean a double composite arrangement. The last digit in the optimization code indicates which gears were constrained to be equal and corresponds to the table used in Fig. 2. Thus 2-DC-1 would correspond to the type XI double composite arrangement with the constraints that $d = j$ and $e = g$ (from Fig. 2) using the speed diagram of Fig. 3(b).

The smallest arrangements are 1-C, 1-SC-7, 2-C and 2-SC-7 which all have a function value of 5.2411. Since 4 solutions are equally small the choice of the arrangement would be based on an additional parameter. Since the single composite arrangements require 1 less gear they would be the choice if no other specifications were to be met. Still, 2 single composite arrangements have the same function value. Since the difference between these two arrangements is their speed diagram, the decision should be based on which speed diagram is most applicable to the problem. Note that the smallest double composite arrangement is given by 2-DC-1 with a value of 7.7833. Clearly, if radial dimensions are most important, double composite arrangements are not best.

Overall the results show that there are many different arrangement capable of satisfying a simple objective function such as minimum radial dimension and that no 1 type of arrangement or 1 type of speed diagram is best for all applications. As many of the arrangements have values close to the optimum, additional constraints can be added to narrow the choice. A dynamic parameter was developed for this.

6. DYNAMIC CRITERIA

The gear train must be designed so that it operates away from the torsional natural frequencies or else dangerous vibrations will be present.

The torsional natural frequencies depend on the inertia's and stiffness of each of the components as well as the way in which they are connected. For a 9 speed gear train, each speed represents a different system because the components are coupled differently. Thus 9 separate analyses are required to find the torsional natural frequencies. Each of these 9 possibilities will be a multi-degree of freedom branched torsional system. The solution of such problems is possible using a number of different techniques, such as transfer matrices, however, considerable time is required.

To solve this type of problem at each iteration of a kinematic optimization to ensure that the natural frequencies are satisfactory is extremely inefficient. Instead, a criteria was developed that could be easily evaluated during the kinematic optimization to indicate if an arrangement would likely be good or bad dynamically.

The dynamic properties of a gear train are closely related to the torsional stiffness, thus an increase in this stiffness will generally improve the dynamic properties. Marchelek[3] has shown that the elasticity or a gear train, which is the inverse of stiffness, is the sum of the elasticity of the connected components (all components running at the same speed). Components running at different speeds must be normalized, based on energy requirements by multiplying the elasticity by the square of the speed ratio.

$$e_e = \sum_{j=1}^{n} (e_j)(N_{sj}) \tag{45}$$

where

$$N_{sj} = n_s/n_j. \tag{46}$$

From eqn (45) it is clear that the smaller the value of N_{sj} the smaller the elasticity and the greater the stiffness (up to a certain point where the increased mass of larger gears counters the benefit).

The gear ratios are defined as soon as the kinematic arrangement is known. Thus a criteria based on avoiding excessive gear ratios would be easy to impliment and reasonably effective.

7. THE OBJECTIVE FUNCTION

The total volume of the gearbox is equal to the height x width x length or

$$V = (H)(W)(L). \tag{47}$$

The height is the sum of the radial dimension across the gears. The width is represented by the diameter of the largest gear. The length depends on whether a single composite, double composite or conventional arrangement is used and the width of the gears.

From the earlier analysis it is clear that the static stiffness is increased if the gear ratios are small. Thus, a measure of the dynamic characteristic can be based on the sum of the largest gear ratios in each mesh.

$$D = \sum_{i=1}^{\#\text{ groups}} (r_{\max})_i. \tag{48}$$

If composite arrangements are used, shorter shafts are required again increasing the static stiffness. Thus a modifying parameter is used to include this benefit. For a conventional arrangement the modifying parameter γ is equal to 1. For a single composite, γ equals 0.95 and for a double composite, γ equals 0.90. Thus

$$D^* = \gamma D. \tag{49}$$

The volume function and the dynamic characteristic function can be combined in a weighted objective function to provide the necessary optimization strategy. Thus the goal is

$$\underset{(Y_1, Y_2, Y_3, Y_4)}{\text{Minimize}} (\alpha V + \beta D^*) \tag{50}$$

Table 3. Optimization summary—selected results for weighted objective functions

Phi	Weighting Factors Size	Stiffness	Type	Height	Volume	Stiffness	Objective Function Value	Rating
1.260	1.0	0.0	2-SC-7	4.26	93.8	265.4	93.78	1
			1-SC-7	4.26	93.8	265.5	93.78	2
			2-C	4.26	119.4	279.4	119.35	17
			1-C	4.26	119.4	279.5	119.36	18
			2-DC-1	6.60	196.1	173.3	196.05	21
			1-DC-4	7.68	271.3	602.6	271.32	22
	0.5	0.5	1-SC-9	5.88	186.4	109.0	147.67	1
			2-SC-9	5.48	150.1	148.3	149.22	5
			1-C	5.33	182.8	155.2	169.02	11
			2-C	5.28	188.3	157.5	172.90	12
			2-DC-1	6.90	198.3	160.8	179.51	16
			1-DC-4	7.68	271.3	602.6	436.96	23
	0.0	1.0	2-SC-9	17.01	2156.2	35.2	35.21	1
			2-C	11.21	271.3	37.1	37.07	10
			1-C	10.60	927.2	37.1	37.07	11
			1-SC-6	9.78	666.9	40.2	40.20	12
			2-DC-2	16.62	1759.6	42.8	42.81	14
			1-DC-4	7.68	271.3	602.6	602.60	25
1.414	1.0	0.0	1-SC-7	5.24	163.0	474.7	163.03	1
			2-SC-7	5.24	163.0	474.5	163.04	2
			2-C	5.24	207.5	499.7	207.49	13
			1-C	5.24	207.5	499.7	207.51	14
			2-DC-1	7.78	272.4	413.4	272.37	21
			1-DC-4	9.95	482.0	1093.8	481.98	22
	0.5	0.5	1-SC-8	7.10	312.4	217.1	264.78	1
			2-SC-9	6.76	280.3	250.9	265.58	3
			1-C	6.56	326.7	285.7	306.21	12
			2-C	6.43	332.8	284.9	308.87	13
			2-DC-1	8.37	285.8	336.7	311.27	14
			1-DC-4	9.95	482.0	1093.8	787.90	23
	0.0	1.0	2-DC-5	21.84	3292.1	126.6	126.62	1
			2-SC-8	14.45	908.2	129.0	128.96	3
			1-SC-9	10.23	652.6	129.0	129.00	11
			2-C	11.72	1025.4	135.8	135.77	14
			1-C	10.73	938.9	135.8	135.78	15
			1-DC-4	9.94	482.0	1093.8	1093.77	22

Note: Type identifies arrangement. First digit refers to speed diagram (figure 3). Letters are as C - Conventional, SC - Single Composite, DC - Double Composite. Last digit specifies particular SC or DC layout (figure 2). Height is wrt reference gear a. Volume is wrt a^2 x gear thickness. Rating is out of a possible 38 arrangements.

Table 4. Optimal arrangements

Phi	Weighting Factors Size	Weighting Factors Stiffness	a	b	c	d	e	f	g	h	j	k	l	m
1.250	1.0	0.0	1.00	1.50	2.00	2.00	1.50	1.00	1.00	1.13	1.26	1.26	1.13	1.00
	0.5	0.5	1.00	1.18	1.37	2.62	2.45	2.26	1.00	1.59	2.26	2.88	2.29	1.63
	0.0	1.0	1.00	1.72	2.67	5.00	4.29	3.33	2.31	2.78	3.33	11.52	11.05	10.50
1.414	1.0	0.0	1.00	1.21	1.41	1.41	1.21	1.00	1.00	1.91	2.83	2.83	1.91	1.00
	0.5	0.5	1.00	1.27	1.57	2.64	2.37	2.07	1.00	2.07	3.33	4.00	2.93	1.67
	0.0	1.0	1.00	2.17	3.69	5.00	3.83	2.31	3.83	5.00	6.37	15.08	13.91	12.54

The parameters were constrained to be larger than zero and the smallest allowable gear ratio was selected to be 0.20. Additional constraints could easily be added if desired.

A simple uni-variate random optimization technique was used to obtain results using the logic shown in Fig. 7.

8. RESULTS

Results are shown in Table 3 for two values of phi (1.26, 1.414). The objective function was weighted in three ways: 100% volume, 50% volume and 50% stiffness, 100% stiffness. Results are listed in Table 4 for the best conventional, single composite, and double composite arrangements for each of the possible speed diagrams.

9. CONCLUSIONS

This paper presents a new approach to the optimum design of multispeed gear trains. The general gear diameter equations developed allow the designer to study each of the feasible arrangements and speed diagrams for a given problem in order to find a more global optimum.

This paper also suggests the concept of developing a dynamic parameter that gives an indication of the system dynamics without performing detailed calculations. The dynamic parameter, defined in this paper, works most effectively when combined with the volume function. For 100% stiffness the constraints tend to govern the solutions for the conventional and single composite arrangements.

From the results, the single composite arrangements provides the best solution most frequently. Conventional arrangements tend to be wasteful while the double composite arrangements are very constrained and hence produce few optimums.

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NOMENCLATURE

- a diameter of reference gear
- $b, c, \ldots, m$ gear diameters
- e_e equivalent elasticity
- n_j speed of shaft j
- n_s speed of shaft s
- p range exponent
- r gear ratio
- r_{max} maximum gear ratio of group
- w number of gear pairs in group
- z number of speeds
- D dynamic optimization parameter
- D^* modified dynamic optimization parameter
- H height
- L length
- N_{sj} speed ratio of shaft s wrt shaft j
- S lowest speed ratio
- S_0 output shaft
- V volume
- W width
- Y_i diameter of smallest driving gear in a group
- Y_0 diameter of largest driven gear in a group
- $Y_1, \ldots, Y_4$ normalized independent variables
- d_{pinion} pinion diameter
- d_{gear} gear diameter
- e train value
- γ dynamic arrangement factor
- α, β weighting factors, volume, dynamic parameter respectively
- ϕ step ratio

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CONCEPTION OPTIMALE DES TRAINS D'ENGRENAGES AUX VITESSES MULTIPLES

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Résumé - Cette communication fait l'exposé d'une méthode efficace d'analyse des trains d'engrenages à vitesses multiples et en montre l'utilisation.

Cette technique nouvelle permet d'obtenir l'équation du diamètre de tous les engrenages utilisés dans la transmission, à partir d'informations contenues dans le diagramme des vitesses. Ces équations sont d'une forme telle qu'elles peuvent être générées automatiquement par ordinateur. En outre, ces équations s'appliquent et à l'arrangement général, et aux arrangements composites, simples et doubles.

Il en résulte qu'il est facile d'étudier tous les arrangements cinétiques prometteurs possibles pour un nombre donné de vitesses. La proposition de contraintes diverses avec des fonctions objectives diverses permet l'examen facile de l'influence réciproque des différents paramètres. On démontre la technique au moyen d'un cas particulier de train d'engrenages à 9 vitesses. On utilise une technique d'optimisation à paramètres multiples pour résoudre 19 arrangements différents pour une fonction objective pondérée, à volume minimal et à rigidité maximale.